Origin of Quantum Chromodynamics

Thitipat Sainapha

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Overview

1. **Strong Interaction**
   - Yukawa effective theory
   - Quark model
   - $e^- p^+$ Scattering

2. **Construct the QCD**
   - Renormalization group flow
   - Asymptotic freedom
Before 1930s, there are only 2 fundamental forces in nature: Gravity and Electromagnetism.

Gravity are described by Newton’s law of gravitation, modified by Einstein’s general relativity

Electromagnetism are studied through Maxwell’s theory, also successfully quantized becoming quantum electrodynamics (QED)
At the beginning of 20th century, there are several discoveries about radioactivity and atomic structure.

There should exist other kinds of forces: One binds nucleons inside nucleus while another explains how nucleus radiates high energy $e^-$, known as the famous nuclear $\beta$-decay.

$\beta$-decay can be effectively understood by 4-Fermi theory and its UV complete theory, electroweak theory.
Yukawa theory (III)

The last type was found to be very strong coupled with respects to all other forces, this is why it called the strong interaction.

As 4-Fermi theory in weak interaction case, strong interaction is effectively described by Yukawa theory

\[ \mathcal{L}_{Yukawa} = \mathcal{L}_{Fermion}(\psi) - \frac{1}{2} \pi(\Box + m^2)\pi - g\pi \bar{\psi}\psi \]  

(1)

where \(\psi\) is some hadron such as proton and neutron, whereas \(\pi\) is scalar field known as pion field.
Steal technique from QED, the force between two hadrons (nucleons) determined by exchanging of pion, pictorially represented by t-channel Feynman diagram.

Using Feynman rule to find the relevant truncated S-matrix (each vertex gives coupling constant $g$ up to factor $i$, and internal line gives bosonic Feynman propagator).
We end up with something like

$$\sim -g^2 \frac{i}{k^2 - m^2 + i\epsilon} \quad (2)$$

To get ordinary potential in classical theory, we will calculate the Fourier transformation of (2) in static limit (Born approximation)

$$V(r) = g^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot r}}{-(k^2 + m^2) + i\epsilon} \quad (3)$$
After working with complex analysis, the integral (3) can be done by writing the contour and using Cauchy’s residual theorem, we obtain

\[ V(r) = -\frac{g^2}{4\pi r}e^{-imr} \]  \hspace{1cm} (4)

The form in (4) is formally known as **Yukawa potential** which was used to understand the binding energy inside nucleus. We see from (4) that strong force has short-distance behavior. Note that in \( m \to 0 \) limit, this expression reduces to long distance Coulomb-like potential.
This model was very successful to describe strong interaction. This led Yukawa-san received the Nobel prize in 1949

This model is okay if we think that hadrons like proton and neutron are elementary particle. It’s not!
Gell-Mann and Zweig independently studied SU(3) group, they found mathematically that both mesons and baryons lie on SU(3) Cartan diagram.

Figure: Eightfold way retrieved from https://en.wikipedia.org/wiki/Eightfold_Way_(physics)
Baryons can be constructed by $3 \otimes 3 \otimes 3$ representation while mesons can be constructed $3 \otimes \bar{3}$ representation.

This gives us the hint for baryons and mesons’ substructure so-called quarks

However, why should believe that SU(3) group theory really describes hadrons?
Quark model (III)

Figure: Baryons decuplet retrieved from https://en.wikipedia.org/wiki/Eightfold_Way_(physics)
Historically, we didn’t discovered $\Omega^-$ in the diagram in previous slide at that time. The later discovering of these particle ensured the correctness of eightfold way led Gell-Mann and Zweig recieved the Nobel prize in 1969.

More importantly, the diagram in the previous slide is for spin 3/2 baryons. Although baryons are fermion, the 3/2 spin baryons seem not to support this fact.

This implies that there should be hidden anti-symmetric degree of freedom called color proposed by Nambu.
The spectra of strong interacting particles, in fact, are much more strange than we expected. If we plot graph between spin with mass square. We will find linear behavior known as **Regge’s trajectories**

**Figure:** Regge’s slope retrieved from
http://insti.physics.sunysb.edu/~siegel/reggepart/reggepart.html
The effective model to understand the Regge’s trajectories is there exists string-like structure gluing them together. This historically was the origin of string theory as the theory of strong interaction. Presumably, this is the reason why many people, like Nambu and Gross, who established strong interaction also contributed in string theories.

This assumption currently has changed a lot: The string-like structure is non-Abelian electric flux tube understood in context of dual superconductor model with monopoles.
The another weird phenomenon of strong interaction comes from electron-proton scattering.

In 1911, Rutherford and his team did very famous gold foil experiment, they used classical scattering theory to estimate the cross section formula.

His cross section need to be more precise by working with more accurate scattering process. Such a process need the smallest nuclei and the smallest probe as possible. That are proton and electron.
First, we will pretend that proton is an elementary particle. By this naive treatment, this scattering process can be studied by QED

\[
\begin{align*}
\gamma & \quad e^- \\
p^+ & \quad e^- \\
p^+ & \quad e^-
\end{align*}
\]

As we have seen from Yukawa theory, in massless internal propagator case (photon is massless), this graph will give Coulomb-like potential. Thus, we call this graph the **Coulomb scattering**.
To get quantum correction, we will modify the Feynman of QED vertex to be proportional to generalized QED vertex function $\Gamma^\mu$ defined as

$$
\Gamma^\mu(q) = F_1(q^2)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m_p}q_\nu F_2(q^2)
$$

where $F_1$ and $F_2$ are called form factors. Note that the reason why we can decompose general vertex function this way is suggested from Gordon’s identity. At leading order, $F_1(0) = 1$ and $F_2(0) = 0$. On the other hands, at 1-loop order, $F_2(0) = \frac{\alpha e}{2\pi}$, this gives quantum correction to magnetic moment called anomalous magnetic moment. (Dirac’s equation predicted the gyro-magnetic ratio is 2 while 1-loop QED predicted $2 + \frac{\alpha e}{\pi}$.)
By this parametrization, we can compute the tree-level elastic Coulomb scattering cross section (in lab frame, since \(m_p \gg m_e\), we can think that only electron travels but proton stays at rest)

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha_e^2}{4E^2\sin^4\frac{\theta}{2}} \frac{E'}{E} \left[ \left( F_1^2 - \frac{q^2}{4m_p^2} F_2^2 \right) \cos^2\frac{\theta}{2} - \frac{q^2}{2m_p^2} (F_1 + F_2)^2 \sin^2\frac{\theta}{2} \right]
\]

(6)

known as **Rosenbluth formula**.
The formula (6) may fit very well with experiments but not in all regime. Obviously, it is because proton is not an elementary particle but composite particle containing quarks.
As we have already mentioned that in low energy limit, QED approximation works well. However, in very high enough energy, electron can collide with proton hard enough so that proton break apart into jet of quarks. Then, traditional parametrization will no longer be able to use. This is known as Deep inelastic scattering (DIS).

To fix this, we need to understand what really happened at collision region.
Feynman suggested that the collision of proton or other hadrons can be related to the collision of things inside that hadron (valence quarks, sea quarks and gluons). He call it **parton**.

The partonic momentum is related to proton’s momentum as

\[ p_i^\mu = x P^\mu. \]  

(7)

where \( p_i^\mu \) is 4-momentum of parton of species \( i \), \( P^\mu \) be proton’s 4-momentum and Bjorken value \( x \).
The amazing fact about parton model is that form factors, also cross section, are approximately independent of energy scale we are doing experiments. This behavior is also widely known as Bjorken scaling.

Actually, we will discuss about these phenomenon as the strong constraint for constructing the quantum theory of strong interaction soon!
$e^- p^+ \text{ scattering (IX)}$

**Figure:** Bjorken scaling retrieved from [https://www.science20.com/a_quantum_diaries_survivor/wolf_prize_in_physics_awarded_to_james_bjorken-152846](https://www.science20.com/a_quantum_diaries_survivor/wolf_prize_in_physics_awarded_to_james_bjorken-152846)
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2. **Construct the QCD**
   - Renormalization group flow
   - Asymptotic freedom
Before we go through the problem mentioned in previous section, let’s study about the useful ingredients parallely discovered first.

At the beginning of QED era, people got very confused about the problem of divergences appearing at all loop level QFT calculation. For example, the quantum correction of Coulomb potential turned out to be infinite, so the shift of energy level between 2s orbital and 2p orbital is also infinite! Infinite Lamb shift!
Many people including Dirac himself threw QED into trash bin, waiting until Feynman, Schwinger and Tomonaga to rescue it. The key idea is they argued that infinities are not observable, also all parameters such as mass, coupling constant, field. The observables must be finite. Thus, the parameters appearing in the Lagrangian all have infinity in them. To get observable, all of that parameters must be renormalized.

Suppose we have any operator unrenormalized $O_0$ called bare operator, we can renormalize this operator as

$$O = Z_O O_0 = (1 + \delta_O) O_0$$ (8)
That implies any observable parameter can be thought as its bare parameter in different scale.

This leads us to relate to value of parameter at two different scales determined by first order differential equation known as renormalization group equation, discovered firstly by Gell-Mann and Low.

As in Hamiltonian mechanics, first order differential equation can be used to understand the flow of variable in phase space-like structure. (How the name renormalization group flow or RG flow coming from)
Let’s consider the pure Yang-Mills theory with Lagrangian density

\[ \mathcal{L}_{YM} = -\frac{1}{2} Tr[(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])^2] \] (9)

This Lagrangian is singular because of gauge redundant degree of freedom. Definition of Hamiltonian is then not unique. This non-uniqueness makes us impossible to quantize the theory. To fix this, we need to add extra term to fix gauge sometimes known as Fermi gauge fixing term \(-\frac{1}{\xi} Tr[(\partial_\mu A_\mu)^2]\) with non-dynamical field \(\xi\) so-called Nakanishi-Lautrup auxiliary field.
Since YM is non-Abelian gauge theory, additional term makes Jacobian of path integral non-trivial. The cost of non-Abelian gauge fixing is we need to introduce more fields which are scalar field but satisfy Fermi-Dirac statistics, $c$ and $\bar{c}$ says.

The existence of this field violates spin-statistics theorem and make theory becoming non-unitary; hence, this fields are called Faddeev-Popov ghost and anti-ghost, respectively.

The existence of ghosts modified YM theory to be

\[
\mathcal{L}_{YM} = -\frac{1}{2} \text{Tr}[(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])^2] - \frac{1}{\xi} \text{Tr}[(\partial_\mu A_\mu)^2] + 2 \text{Tr}[\bar{c}\Box c - g(\partial_\mu \bar{c})[A_\mu, c]]
\] (10)
Note that Lagrangian (10) is no longer invariant under gauge symmetry but two new symmetry born: $U(1)$ ghost symmetry and BRST symmetry.

BRST symmetry forms so-called BRST algebra which is so much similar to superalgebra. These two symmetry led Kugo and Ojima to postulate the strong constraint on how quark are confined. We already discussed the evidences of confinement behavior in context of Regge’s trajectories. (Regge’s slope implies that potential energy somehow is linear proportionality and no quark can be escape from this potential!)
Again all parameters in Lagrangian (10) are infinite. We need to renormalize all of that things. In pure YM, there are no infrared cutoff (IR divergence are not regularized since generally fermion’s mass will act as IR regulator). We need to perform the renormalized at non-zero Euclidean energy scale. Namely, $p^2 = -\mu^2$

We define bare gluon propagator to be

\[
D_0(-\mu^2)_{\mu\nu}^{ab} = \frac{i}{\mu^2} Z_3 \left( g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2} \right) \delta_{ab}
\]  

(11)
and bare ghost propagator

\[ G_0(-\mu^2)_{\mu\nu}^{ab} = -i \frac{\mu^2}{\mu^2} \tilde{Z}_3 \delta_{ab} \]  \hspace{1cm} (12)

Also define charge renormalization constant \( Z_1 \) in term of bare three-point vertex function

\[ \Gamma_0^{abc}(p, q, r)_{\mu\nu\rho} = Z_1^{-1} \Gamma^{abc}(p, q, r)_{\mu\nu\rho} \]  \hspace{1cm} (13)
For ghost vertex, we have

\[ \tilde{\Gamma}_{0}^{abc}(p, q, r)_{\mu\nu\rho} = \tilde{Z}_1^{-1}\tilde{\Gamma}^{abc}(p, q, r)_{\mu\nu\rho} \]  \hspace{1cm} (14)

According to Ward-Takahashi identity, these identity gives the relation among renormalization constants (11)-(14)

\[ \frac{Z_3}{Z_1} = \frac{\tilde{Z}_3}{\tilde{Z}_1} \]  \hspace{1cm} (15)
Ward-Takahashi identity implies that the way we renormalize the gluon propagator is the same way we renormalize ghost. Thus ghost is harmful and useful at the same times, one can compute the 1-loop gluon vacuum polarization contributions, only gluon loop amplitudes alone can't be gauge invariant. After we add ghost loop contribution, the total amplitude becomes manifestly gauge invariant!
The last relevant renormalization factor comes from longitudinal part of propagator

\[(D_0^{-1})^{ab}_{\mu\nu} = \ldots + \frac{i}{\xi_0} k_\mu k_\nu, \quad (16)\]

This longitudinal mode is unphysical since this mode is proportional to 4-momentum of gluon corresponding to pure gauge degree of freedom. Thus, after quantization procedure, we can gauge away this degree of freedom by choosing gauge fixing choice such that $\xi = \xi_0 = 0$ called Landau gauge.
Renormalization group flow (XII)

From all renormalization factors we defined previously, we can extract the factors for renormalized wave functions and renormalized coupling constant

\[
A^a_\mu = Z_3^{-1/2} (A_0^a)_\mu \\
c^a_\mu = \tilde{Z}_3^{-1/2} (c_0^a)_\mu ,
\]

(17)

and

\[
g = Z_3^{3/2} Z_1^{-1} g_0
\]

(18)
Suppose our interests is any 1-particle irreducible contribution (1PI) containing any numbers and any kinds of fields and parameters. However, The bare 1PI Green’s function will independent of change of energy scale because bare variables are fixed at only some scale (scale of Lagrangian density). Mathematically,

\[
\mu \frac{\partial}{\partial \mu} \Gamma_0^{(n)}(\Lambda, g_0) = 0,
\]

(19)

Using chain’s rule, we finally obtain

\[
\left[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - n\gamma(g) \right] \Gamma^{(n)}(g, \mu) = 0
\]

(20)

Equation (21) is the Gell-Mann-Low renormalization group equation we were talking about.
Each new functions are defined by

$$\beta(g) \equiv \mu \frac{\partial g}{\partial \mu}$$  \hspace{1cm} (21)

$$\gamma(g) \equiv \frac{1}{2} \mu \frac{\partial \ln(Z_3)}{\partial \mu}$$  \hspace{1cm} (22)

The function (23) are called **anomalous dimension** determined the deriation of conformal scaling, e.g. if we do classically conformal transformation $p \to \lambda p$. In quantum level, 1PI Green’s function will change by classical scaling with extra factor proportional to gamma function.
More important differentiable function is $\beta$ function because it is determined the change of coupling constant as energy scale changing. Thus, the running of coupling encoded inside these powerful function. There are 3 important case depending on value of $\beta$.

- $\beta > 0$ coupling constant increases as increasing of energy scale.
- $\beta = 0$ theory is scaleless, e.g. conformal field theory.
- $\beta < 0$ coupling constant decrease as increasing of energy scale.
One might naturally ask that why case of running coupling of strong interaction should be?

The answer can be obtained from the hints specifically Bjorken scaling behavior. Bjorken scaling is occurred at high energy deep inelastic scattering (experiment done by SLAC). Bjorken scaling implies the running of cross section (determined by Gell-Mann-Low equation) must be very small. Alternatively, $\beta(g_\infty) \approx 0$ with $g_\infty$ coupling at ultra high energy level. Finally, we reach the fact the strong interaction must be UV stable or asymptotically free!
Coleman and Gross proved that no theory without non-Abelian gauge field can be asymptotically free. All we have to do is just calculate the 1-loop renormalization constant of Yang-Mills theory.

\[ Z_3 = 1 + \frac{1}{\varepsilon} \frac{g^2}{16\pi^2} \left( \frac{13}{3} C_A - \frac{8}{3} N_f T_F \right) \]  

(23)

For ghost vacuum polarization, we get

\[ \tilde{Z}_3 = 1 + \frac{1}{\varepsilon} \frac{g^2}{16\pi^2} \left[ \frac{3}{2} C_A \right] \]  

(24)
On the other hand, $Z_1$ can be found through calculation the 3-point function

$$Z_1 = 1 + \frac{1}{\varepsilon} \frac{g^2}{16\pi^2} \left(-\frac{3}{2} C_A\right)$$

Finally, $\tilde{Z}_1$ is not necessary to compute because we can calculate alternatively by using Ward-Takahashi identity (16)
Solving for $\beta$ function by

$$\beta(g) = -g\mu \frac{\partial}{\partial \mu} \left( \frac{Z^{3/2}}{Z_1} \right)$$  \hspace{1cm} (26)$$

After long calculation, we can solve perturbatively to get 1-loop beta function

$$\beta(g) = -\varepsilon g - \frac{g^3}{16\pi^2} \left( \frac{11}{3} C_A - \frac{4}{3} N_f T_F \right)$$  \hspace{1cm} (27)$$

where $\varepsilon$ is UV cut off setting to be zero after ending the full calculation. $C_A$ is Casimir operator of adjoint representation defined to be $C_A = C(\text{Adj})$, $d(R)C(R) = T(R)d(G)$ and $Tr[T_R^a T_R^b] = T(R)\delta^{ab}$. 
Asymptotic freedom (V)

For SU(N) gauge theory. $d(G) = N^2 - 1$, $T(Fund) = \frac{1}{2}$ and $T(Adj) = N$. We can easily get the full expression of beta function in SU(N) gauge theory

$$
\beta(g) = -\frac{g^3}{16\pi^2} \left( \frac{11}{3} N - \frac{2}{3} N_F \right). \tag{28}
$$

We found that for the case that $N > \frac{2N_f}{11}$, the theory will be asymptotically free.
Our current knowledge is number of flavors $N_f = 6$, then for SU(N) for $N$ greater than 1, everything will be fine. This also implies Coleman and Gross proof. Then what $N$ should it be? Since we know that color degree of freedom has been discovered. For baryon decuplet state to be anti-symmetric, it should be color singlet. According to group theory, the only possible way that 3 baryons can have color singlet is color symmetry should belong to SU(3)

The discovering of asymptotic freedom led to the origin of quantum theory of strong interaction known as quantum chromodynamic (QCD). Thus, this discovering due to the work of David Gross and his advisee Frank Wilzcek is worth the Nobel prize in 2004! 30 years after published their paper.
References


The End