Review on Higgs Mechanism without Spontaneous Symmetry Breaking

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Abstract

In this article, I decided to review the knowledge earned from reading the literature about the gauge-independent Higgs mechanism proposed by Kondo. In the present article, I will focus on reviewing the paper [22, 23] to understand the Higgs mechanism without concerning the spontaneous symmetry breaking.

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1 Introduction

It cannot be refused that the hearts of the high energy physics are symmetry and symmetry breaking. The spontaneous symmetry breaking of the global continuous
symmetry can successfully describe the pion physics while the spontaneous symmetry breaking in the gauge theory is the suggestive mechanism to be used to understand the mass generation of the W and Z bosons in the electroweak model. In the QCD sector, the dynamical generation of the gluon mass gap and the confinement effect is a fundamental feature. It will be promising if we can do the same thing in QCD physics. However, the so-called Higgs mechanism is dependent strongly on the gauge fixing condition used in the considering theory. However, the confinement and the dynamical generation of the mass gap are expected to be the gauge-independent mechanism. Moreover, in the QCD, there is no scalar field that responds for spontaneous symmetry breaking. To solve these mentioned problems, we have to create the Higgs mechanism or the generation of the mass of the gauge field without breaking any symmetry transformation even in the vacuum level. The plan of this article is as follows: starting with discussing the original way to describe the Higgs mechanism that relies on the spontaneous symmetry breaking in an Abelian gauge theory, then we will explore the blind spot of the traditional version of the Higgs mechanism on the theoretical ground. After that, we will suggest the possible physical interpretation of the result of the mechanism. Then, in the next right section, we will consider the non-Abelian gauge theory instead. There we will repeat almost all we have done in an Abelian case. Finally, we will discuss the feature that implies the confinement phenomenon in QCD physics and explain the way to eliminate the scalar field’s degree of freedom to become the pure Yang-Mills theory.

2 Abelian-Higgs Model

We will begin by discussing the Higgs mechanism with the original treatment concerning the spontaneous symmetry breaking. To understand and gain a basic principle about this idea, we will restrict ourselves to the simpler case that is an Abelian version of the Higgs mechanism. The simplest but non-trivial model which is suitable to be studied to understand the spontaneous symmetry breaking is the complex scalar field with the appropriate form of the potential term. The corresponding Lagrangian takes the form

\[
\mathcal{L} = (\partial_\mu \phi)^\dagger \partial^\mu \phi - V(\phi, \phi^\dagger),
\]  

(1)
where the potential, in the problem at hand, is chosen to be the Mexican hat-shape

\[ V(\phi, \phi^\dagger) = \frac{\lambda}{2} (\phi^\dagger \phi - v^2)^2. \] (2)

Observe that the full Lagrangian possesses the global phase transformation or the \(U(1)\) global symmetry acting on the scalar field and its Hermitian conjugation as

\[ \phi \rightarrow e^{i\alpha} \phi, \quad \phi^\dagger \rightarrow \phi^\dagger e^{-i\alpha}. \] (3)

The main difference of the spontaneous symmetry breaking from the explicit version of the symmetry breaking effect is the total action still preserves the corresponding symmetry. However, the \textbf{spontaneous symmetry breaking} concerns the broken of the symmetry at the vacuum level. To understand this statement, we have to first compute the vacuum expectation value of the field. Since the kinetic term of the Lagrangian is positive-definite, the vacuum state corresponds to the state \(|\Omega\rangle\) that minimizes the potential term. Thus, the expectation value of the field can be computed by solving the following equations

\[ V'(\langle \phi \rangle, \langle \phi^\dagger \rangle) = 0. \] (4)

It is easy to show that the expectation value of the field at the vacuum can be expressed as

\[ \langle \phi \rangle = ve^{i\varphi}, \] (5)

where \(\varphi\) is an arbitrary phase factor. We can see precisely that there are infinitely degenerate vacua (we will understand soon enough why they are degenerate). When we study the dynamics of the field in one specific vacuum, the symmetry turns out to transform from this considering vacuum into another. Thus, the dynamics of the field in the vacuum state does not preserve under the underlying \(U(1)\) symmetry.

The way to understand the dynamics of the field in the vacuum state is to perform the perturbation around the vacuum state and study the behavior of the fluctuating fields. Since we have a freedom to focus on only one specific vacuum state due to the presence of the global symmetry, we choose to work in the state such that the field expectation value is purely real, namely, we have

\[ \langle \phi \rangle = v. \] (6)
Thus, we can express the field around this vacuum as

\[ \phi(x) = (v + \sigma(x))e^{i\pi(x)F_{\pi}}, \]  

(7)

where \( F_{\pi} \) is a mass dimension one parameter introduced to make the angular part of the scalar field \( \pi(x) \) a right canonical dimension. Plug this expression (7) into the Lagrangian (1), we end up with

\[ L = \partial_\mu \sigma \partial^\mu \sigma + \frac{(v + \sigma)^2}{F_{\pi}^2} \partial_\mu \pi \partial^\mu \pi - 2\lambda v^2 \sigma^2 + \ldots, \]  

(8)

where \( \ldots \) refers to the self-interacting term between the radial part and angular part of the perturbation. This effective model is known as the **linear sigma model**.

Observe that this model has two important fields. One of them is the radial perturbation or currently known as the **Higgs field** \( \sigma \) (historically, called the sigma field) with the mass \( m_\sigma = 2v\sqrt{\lambda} \). The remaining field is the angular perturbation \( \pi \) or called **pion** by historical reason. Notice also that the pion field has no rest mass. Intuitively speaking, the pion is impossible in principle to have a mass term since the global \( U(1) \) symmetry forces the pion to respect the shift symmetry \( \pi(x) \to \pi(x) + F_{\pi}\alpha \) accordingly. In particular, the existence of the massless pion field is a special feature in the theory with a spontaneously broken continuous symmetry. This special feature is called the **Goldstone theorem** [14]. To sketch the proof of this theorem, we essentially recall what will happen when the total action respects the continuous symmetry. The so-called **Noether’s theorem** states that if the action is preserved under the continuous symmetry, there will always exist the (on-shell) globally conserved current \( J^\mu \) associated with such a symmetry transformation. In the 4-vector language, the conservation law is equivalent to the continuity condition

\[ \partial_\mu J^\mu = 0. \]  

(9)

It can be checked simply that this condition implies the conservation of the so-called **Noether’s charge**

\[ Q \equiv \int d^3x \, J^0(x), \]  

(10)

i.e. \( \frac{dQ}{dt} = 0 \) up to the boundary term at the spatial boundary. In the Hamiltonian formalism, the conservation of something means that its Poisson bracket with the Hamiltonian yields zero, or becoming the Lie bracket after quantization. This is
the reason why the states related through the symmetry transformation generated
by the Noether’s charge \( Q \) carry the same amount of energy or novelly said to be
degenerate. Explaining explicitly, suppose that the vacuum state carries \( E_0 \) energy,
i.e. \( H|\Omega\rangle = E_0|\Omega\rangle \), it can be shown that the transformed state carries the same
energy if the Noether’s charge commutes with the Hamiltonian operator

\[
H(Q|\Omega\rangle) = QH|\Omega\rangle = E_0(Q|\Omega\rangle). \tag{11}
\]

If the spontaneous symmetry breaking occurs, the Noether’s current can not anni-
hilate the vacuum state since the evolution operator constructed by exponentiating
the Noether’s charge can not act on the vacuum state as the identity operation. Thus, we can always construct the state of the following form

\[
|\pi(\vec{p})\rangle = \int d^3x \ e^{i\vec{p} \cdot \vec{x}} J^0(x)|\Omega\rangle. \tag{12}
\]

Due to the fact that the limit \( \vec{p} \to 0 \) leads to the state \( |\pi(0)\rangle = Q|\Omega\rangle \) with vacuum
energy \( E_0 \), the kinetic energy of the pion has a massless type. The pion is alterna-
tively called the **Nambu-Goldstone boson**. Here we end the sketch of the proof
of the Goldstone theorem.

Now what if we promote from the global symmetry transformation into the local
symmetry, technically known as gauging. It is precise that the extra term will ap-
ppear in the transformed Lagrangian, to perfectly cancel the additional contribution,
we need to introduce the so-called **gauge field** or **connection** with the appropriate
transformation rule into the partial derivative term. The new Lagrangian becomes

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^{\mu} \phi)^\dagger D_\mu \phi - V(\phi), \tag{13}
\]

where the covariant derivative \( D_\mu \) is defined to be

\[
D_\mu \phi = \partial_\mu \phi - ieA_\mu \phi. \tag{14}
\]

Moreover, the additional first term is the kinetic term of the gauge field by the
gauge-invariant field strength tensor

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{15}
\]

This Lagrangian is known as the **Abelian Higgs model**. Once again, we can repeat
the same step as we have done in the non-gauging situation. First of all, notice that
the gauging procedure does not affect the potential term, thus the minimization of the potential term leads to the degenerate vacua that are the main source of the spontaneous symmetry breaking. Choosing the vacuum state such that the value of the field expectation is a purely real value as earlier, then substituting the perturbation around the vacuum state into the Abelian-Higgs Lagrangian. We have something like

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2 v^2 A_\mu^2 + ...$$

(16)

We see here amazingly that the gauge field becomes massive with the photon mass $m_A = \sqrt{2}ev$. However, it is still complicated to extract the physical spectrum of the field since there exists the complicated cross term between the Higgs field $\sigma$, pion $\pi$, and the photon $A_\mu$. The complication can be simply resolved by the following steps. To begin with, we can eliminate out the $\sigma$ particle from the theory by requiring that the mass of the sigma field $m_\sigma$ is so large that the considering energy scale cannot excite such a mode at all. This regime is sometimes called the decoupling limit which, in fact, changes from the linear sigma model into the non-linear sigma model. In the non-linear sigma model, the sigma field is integrated out from the physical spectrum, the Lagrangian reduces into

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(\frac{v}{F_\pi} \partial_\mu \pi - \frac{1}{\sqrt{2}} m_A A_\mu \right)^2.$$

(17)

This is not fully simplified yet since there is still the combination between the Goldstone mode and the photon. To eliminate the Goldstone mode from the theory, we observe first that what form of the gauge symmetry looks like in the current problem. As we have discussed in the global case once that the pion is nothing more and nothing less but the perturbation along the direction of the gauge rotation, thus the pion must respect the shift symmetry

$$\pi(x) \rightarrow \pi(x) + F_\pi \alpha(x)$$

(18)

In addition, the gauge symmetry also transforms the gauge field as

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x).$$

(19)

Thus, we have the freedom to transform the fields into the configuration such that the $\pi' = 0$ or

$$A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{eF_\pi} \partial_\mu \pi.$$

(20)
Now the Lagrangian of the transformed field configuration depends not on the Goldstone boson any longer. This is precisely the gauge fixing condition which is called the unitary gauge. In this gauge condition, the degrees of freedom of the gauge field becomes three since the Goldstone boson is absorbed into it. That is why the gauge field in such a model becomes massive intuitively. The mechanism that let the gauge field gains its mass is known as the Higgs mechanism \[9, 18\].

The other choice of the gauge fixing constraint is the Landau gauge \( \partial_\mu A^\mu = 0 \). This gauge condition destroys the cross term up to the integration-by-parts. The gauge field in this gauge choice has two degrees of freedom and the rest degree of freedom is the free Goldstone mode. In particular, the extra longitudinal mode of the gauge field in the unitary gauge is precisely the extra pion field in the Landau gauge. This is called sometimes the Goldstone boson equivalence theorem \[5, 28\]. This kind of duality is useful in studying the electroweak model, for example, whenever we want to study the scattering amplitude concerning the W-boson exchange which is sometimes complicated, we can dually change the perspective from the massive vector boson theory into the scattering amplitude concerning the pion interaction.

What do we learn from this careful analysis? This tells us that the way we look to the Higgs mechanism is dependent strongly on the gauge fixing choice. To be precise, the Higgs mechanism from the spontaneous symmetry breaking effect is “gauge-dependent” and can be interpreted physically in the unitary gauge only! The natural question to ask is that can we not fix the gauge and still perform the spontaneous symmetry breaking in the gauge theory? The saddest answer is the gauge fixing procedure is inevitable. The reason is that, unlike its global version, the gauge symmetry is not the symmetry but redundancy in choosing the field configuration. Thus, at first glance, we choose the vacuum state, we have already chosen the gauge choice accidentally.

The answer is also supported by the lattice result. In the lattice gauge theory, there is the Elitzur theorem \[8\] stating that “the gauge symmetry cannot be broken spontaneously without fixing the gauge”. The main idea of the theorem is as follows: If there is no gauge fixing condition, all gauge-non-invariant correlation function vanishes. This implies that the expectation value of the field \( \phi \) vanishes.
due to this argument. Thus, we need to fix the gauge in order to make the vacuum condensation \( \langle \phi \rangle \neq 0 \) which leads to the spontaneous symmetry breaking. It can be further shown that \( \langle \phi \rangle \neq 0 \) only for the Landau gauge and \( \langle \phi \rangle = 0 \) for all other covariant gauge fixing. Moreover, in the unitary gauge, \( \langle \phi \rangle \) turns out to be non-zero no matter what shape of the potential really looks like (remember that there can be the situation in which the shapes of two potentials in the discretized version look the same in the continuum limit). After fixing the gauge degree of freedom, there still exists the global subgroup \( H' \), which cannot be singled out by the gauge constraint, of the gauge symmetry \( G \) called the remnant gauge symmetry [15].

The spontaneous breaking of the remnant gauge group \( H' \) responds mainly for the underlying Higgs mechanism. Unfortunately, the remnant gauge group \( H' \) is not “unique” leading to the distinction inside the phase diagram of the gauge theory. There is also a theorem called the Fradkin-Shenker-Osterwalder-Seiler theorem [12, 24] claiming that each phase can be mapped to one another by analytic continuation. Physically speaking, there is no at all the phase transition among the Higgs phases, separated by the choice of \( H' \), in the phase diagram. In summary, the concept of spontaneous symmetry breaking in the gauge theory is extremely ambiguous.

3 Gauge-Independent Higgs Mechanism in Abelian Theory

So far, we have shown that the spontaneous symmetry breaking in the gauge theory with the suitable gauge condition is sufficient for the Higgs mechanism. However, it is not necessary to generate a Higgs mechanism. It was first suggested by Kondo that the Higgs mechanism or the mass gaining of the gauge field can occur gauge-invariantly. To do so, let’s define the new field variable denoted by \( W_\mu \) from the old fields as follows

\[
W_\mu \equiv e^{-i\phi^i D_\mu \phi}. \tag{21}
\]

where \( \hat{\phi} \) is defined to be a normalized scalar field \( \hat{\phi} \equiv \frac{\phi}{v} \). Suppose further that the theory is subjected by the holonomic constraint of the form

\[
f(\hat{\phi}, \hat{\phi}^\dagger) = \hat{\phi}^\dagger \hat{\phi} - 1 = 0, \tag{22}
\]
which is obviously equivalent to the equation describing the allowed values of the scalar field in the vacuum (equation that tells the shape of the so-called vacuum manifold)

\[
\langle \phi^\dagger \rangle \langle \phi \rangle = |\langle \phi \rangle|^2 = v^2.
\]  

(23)

Observe first that, at this level of the requirement, the gauge symmetry is still respected. Using this constraint, we will obtain the identity by straightforwardly derivative the constraint equation

\[
\hat{\phi}^\dagger \partial_\mu \hat{\phi} = - (\partial_\mu \hat{\phi}^\dagger) \hat{\phi}.
\]  

(24)

It can be used to show further that

\[
\hat{\phi}^\dagger D_\mu \hat{\phi} = \hat{\phi}^\dagger (\partial_\mu - ieA_\mu) \hat{\phi}
= -(\partial_\mu \hat{\phi}^\dagger + ieA_\mu \hat{\phi}^\dagger) \hat{\phi}
= -(D_\mu \hat{\phi})^\dagger \hat{\phi}.
\]  

(25)

With the help of this identity, we can show simply that

\[
(D_\mu \phi)^\dagger D^\mu \phi = v^2 (D_\mu \hat{\phi})^\dagger (\hat{\phi} \hat{\phi}^\dagger) D^\mu \hat{\phi}
= \frac{1}{2} (2e^2 v^2) \left[ \frac{i}{e} \hat{\phi}^\dagger D_\mu \hat{\phi} \right] \left[ \frac{i}{e} \hat{\phi}^\dagger D^\mu \hat{\phi} \right]
= \frac{1}{2} m_W^2 W_\mu W^\mu.
\]  

(26)

As we can see that the new field \( W_\mu \) is a massive field with the same mass as the gauge field in the spontaneously broken gauge theory, i.e. \( m_W = m_A \). However, the huge difference is that the presence of the mass of the \( W_\mu \) field does not violate the gauge symmetry. We can understand directly by observing the action of the gauge group to the expression of the \( W_\mu \). Namely, we have

\[
W_\mu \to W'_\mu = \frac{i}{e} (\hat{\phi}^\dagger e^{-i\alpha}) e^{i\alpha} D_\mu \hat{\phi} = W_\mu.
\]  

(27)

Thus, \( W_\mu \) is complete gauge-invariant since it does not belong to the element of the Lie algebra of the corresponding gauge group. So where is the spirit of the Lie algebra that the gauge field \( A_\mu \) expects to have? This suggests us to decompose the original gauge potential \( A_\mu \) into two parts: the massive mode and the so-called residual mode, i.e.

\[
A_\mu = W_\mu + V_\mu.
\]  

(28)
The residual gauge mode $V_\mu$ transforms in the same way as the full version. To determine the specific form of the residual gauge field, we set the massive gauge field $W_\mu$ into zero, we can see precisely from the definition of the massive field that the following relation holds, whenever $W_\mu = 0$,

$$(\partial_\mu - ieV_\mu)\hat{\phi} = 0. \quad (29)$$

It is not difficult to be solved, we have

$$V_\mu = -\frac{i}{e}(\partial_\mu \hat{\phi})\hat{\phi}^\dagger. \quad (30)$$

In fact, this expression is nothing more and nothing less but

$$V_\mu = A_\mu - W_\mu. \quad (31)$$

However, we need to carefully keep in mind that the theory is subjected to the constraint above. One might ask how is this procedure relates to the Higgs mechanism concerning the spontaneous symmetry breaking. Let’s choose the choice of the gauge fixing such that the vacuum expectation value of the field aligning along its purely real part (equivalent to fix the gauge by using the unitary gauge in the decoupling limit)

$$\phi = v. \quad (32)$$

With this choice of the field, we can see that the massive mode $W_\mu$ reduces itself into the original mode $A_\mu$ and the residual mode vanishes $V_\mu = 0$ (up to the singular topological term). Thus, the result from gauge-independent procedure is ensured to be reduced into those obtained from the traditional method.

Now comes to the main question of physics, what is the physical meaning of the new field variable? Traceback to the original version, we see that the main fields that play roles in the theory are scalar field $\phi$ and the gauge field $A_\mu$. The meaning is pretty clear enough. The scalar field there is interpreted as a composite particle (not elementary particle) created from the condensation of two electrons in a low-temperature regime. This condensate is known as the Cooper pair [4]. The generation of the Cooper pair breaks the symmetry and drives the Higgs mechanism. Now the gauge field interpreted as the external photon or an applied electromagnetic field gains its mass when it moves through the Cooper pair. In such a
material, the electromagnetic interaction becomes short-range. This is the feature of the material called the superconductor.

In the case of the gauge-independent Higgs mechanism, the field contents contain three fields: the massive gauge field \( W_\mu \), the residual gauge mode \( V_\mu \) and the (holonomic) constraint field \( \hat{\phi} \). The Lagrangian is taken from the changing of the variables into

\[
L = -\frac{1}{4} W_\mu W^{\mu\nu} - \frac{1}{4} V_\mu V^{\mu\nu} - \frac{1}{2} W^{\mu\nu} V_{\mu\nu} + \frac{1}{2} m_W^2 W_\mu^2 + \frac{\lambda}{2} f(\hat{\phi}, \hat{\phi}^\dagger) \tag{33}
\]

where we have introduced the field strength tensor for \( W_\mu \) and \( V_\mu \) as \( W^{\mu\nu} \) and \( V^{\mu\nu} \), respectively. We have pushed the responsibility for being dynamical fields into two gauge modes \( W_\mu \) and \( V_\mu \) and forced the scalar field to be non-dynamical, to be precise, the constraint with the Lagrange’s multiplier \( \frac{\lambda}{2} \). Thus, to understand the theory physically, we need to understand the intuitive picture of two gauge fields. With the help of the identity we have derived earlier, we can rewrite the expression of the massive field into

\[
W_\mu = i \frac{e}{2} \left( \hat{\phi}^\dagger \slashed{D}_\mu \hat{\phi} - (\slashed{D}_\mu \hat{\phi})^\dagger \hat{\phi} \right). \tag{34}
\]

Whereas, the residual field takes the same form without the coupling term to the gauge field. For the one who is familiar with this quantity will notice immediately that this is nothing but the conserved Noether’s current associated with the global symmetry (after gauging) up to some constant of proportion. Thus, instead of thinking about the dynamics of both Cooper pair that couples to the external electromagnetic field, the procedure describes the picture of the conserved currents flowing inside the material. We see here also that there are two types of the conserved current, one is current with the short-range flow while the other is the current that can flow infinitely far. However, we need to keep in mind that the model is subjected to the holonomic constraint as expressed above.

4 Non-Abelian Version

Now we move to our main dish, a non-Abelian gauge theory. Speaking briefly, the non-Abelian gauge theory is a straightforward generalization of electromagnetism which is technically a gauge theory with Abelian group \( U(1) \). To be general, let’s
suppose that our gauge theory is described by the non-Abelian group denoted by $G$ meaning that the gauge field can be represented as an element $T^a$ in the vector space spanned by elements of the corresponding Lie algebra $\mathfrak{g}$. Namely, we can express

$$A_\mu = A_\mu^a T^a. \quad (35)$$

The action of the gauge group $G$ on the gauge field takes the form

$$A_\mu \rightarrow U^{-1} A_\mu U + \frac{i}{g} U^{-1} \partial_\mu U, \quad (36)$$

where $U$ is an element of the gauge group $G$. Besides the gauge sector, there is also the scalar field sector which is chosen to be those represented by the adjoint representation of the gauge group since the pattern of the spontaneous symmetry breaking generated from this sort of the scalar field is significantly interesting. What I mean adjoint representation is as follows: let’s write the scalar field as $\Phi$. As same as the gauge field, the scalar field can be expressed as the vector in the Lie algebra. However, the big difference is the action of the gauge group on the scalar field rather takes the form

$$\Phi \rightarrow U^{-1} \Phi U. \quad (37)$$

In the problem at hand, the covariant derivative, along with the spirit of the Lie algebra, becomes

$$\mathcal{D}_\mu \Phi = \partial_\mu \Phi - ig[A_\mu, \Phi]. \quad (38)$$

To express the covariant derivative in the component of the Lie algebra, we use the definition of the Lie algebra which is the algebra among the infinitesimal generators of the Lie algebra, i.e.

$$[T^a, T^b] = if^{abc} T^c. \quad (39)$$

We will see precisely that the covariant derivative takes the following form

$$\mathcal{D}_\mu \Phi^a = \partial_\mu \Phi^a + g f^{abc} A_\mu^b \Phi^c. \quad (40)$$

Now the full Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4T_F} \text{Tr}(F^{\mu\nu} F_{\mu\nu}) - \frac{1}{2T_F} \text{Tr}(\mathcal{D}_\mu \Phi \mathcal{D}^\mu \Phi) - \frac{\lambda}{2} \left( \frac{1}{T_F} \text{Tr}(\Phi^2) - v^2 \right), \quad (41)$$

where $T_F$ is known as the index of the fundamental representation of the group $G$ defined in such a way that the trace over the Lie algebra is chosen to be

$$\text{Tr}(T^a T^b) = T_F \delta^{ab}. \quad (42)$$
If we restrict more to the case of the special unitary group $G = SU(N)$, the fundamental index can be normalized to be $T_F = \frac{1}{2}$. In the traditional perspective of the Higgs mechanism, we can perform the minimization of the potential term of this Lagrangian to investigate the vacuum structure of the theory. After that, we can choose partially the gauge condition, e.g. the unitary gauge, to choose the particular form of the vacuum. In the chosen vacuum, the full symmetry group $G$ is preserved no longer. However, the chosen vacuum will be invariant under the unbroken subgroup $H$. Once again, substitute the fluctuation over the vacuum into the Lagrangian, we will end up with the mass of the non-Abelian gauge field.

As we have emphasized that, we have to choose the gauge condition to choose the specific expectation value of the vacuum-valued field. It is no wonder gauge-dependence mechanism which is, as before, extremely unsatisfied. Thus, we introduce the new field variable $W_\mu$ as

$$W_\mu \equiv -\frac{i}{g}[\hat{\Phi}, D_\mu \hat{\Phi}], \quad (43)$$

where $\hat{\Phi}$ is a normalized scalar field defined in the same way as the Abelian’s situation. Notice that the presence of the commutator is introduced to encode the non-Abelian behavior of the theory. In component, we can write

$$W_\mu^a = \frac{1}{g} f^{abc} \hat{\Phi}^b (D_\mu \hat{\Phi})^c. \quad (44)$$

Observe that the bilinear combination of this new field variable contains the product of two Lie’s structure constants which is dependent strongly on the Lie group behavior, i.e.

$$W_\mu^a W_\nu^a = \frac{1}{g^2} f^{abc} f^{ade} \hat{\Phi}^b (D_\mu \hat{\Phi})^c (D_\nu \hat{\Phi})^d. \quad (45)$$

Some Lie group has a complicated expression of the product between two structure constants while some are less complicated. Therefore we, for simplicity, restrict into the situation that our gauge group is the non-Abelian $SU(N)$. It can be shown that the product of the structure constants in the $SU(N)$ gauge group problem can be rewritten through the following identity [17]

$$f^{abc} f^{ade} = \frac{2}{N} (\delta^{bd} \delta^{ce} - \delta^{be} \delta^{cd}) + d^{bda} d^{cea} - d^{bda} d^{cea}, \quad (46)$$

where $d^{abc}$ is a quantity defined as follows

$$d^{abc} \equiv 2 \text{Tr}(T^a \{T^b, T^c\}). \quad (47)$$
Unfortunately, even we have used the mentioned identity, the final expression is still not simplified. The complication follows from the combination of the symmetric quantity $d^{abc}$. In the simplest possible situation, i.e. the $SU(2)$ non-Abelian gauge $T^a = \frac{\sigma^a}{2}$ where $\sigma^a$ is a Pauli matrix, we can see precisely that

$$d^{abc} = \delta^{bc} \text{Tr}(T^a) = 0. \quad (48)$$

Remember that since the determinant of an element of the special group is, by construction, a unity, thus the infinitesimal generator of such a group is traceless. Intuitively speaking, the physical interpretation of this quantity is that it determines the chiral anomaly structure of the considering gauge group. The $SU(2)$ is known to be the commutator group, i.e. every element in the group can be written as a commutator between two elements in the same group. In the language of the representation theory, the commutator group has only one trivial representation, thus it can be shown that it will be technically anomaly-free. This is the physical reason why the electroweak can be described by the $SU(2)$ gauge group coupled to only the left-handed fermions without producing any gauge anomaly into the standard model.

Since we have decomposed the gauge field and the scalar field in components, thus we can swap the position of them commutatively. It is then not too difficult to show that, in the $SU(2)$ gauge, we end up with

$$W^a_\mu W^a_\mu = \frac{1}{g^2 v^2} (D_\mu \Phi)^a (D^\mu \Phi)^a. \quad (49)$$

Equivalently, we can conclude that the gauge-independent Higgs mechanism in the $SU(2)$ gauge theory is explicit, where $m_W \equiv g v$ (let’s remind that the absence of the coefficient $\sqrt{2}$ comes from the different normalization from the Abelian case).

Unlike the previous case, we can deduce from the explicit expression of the current field $W^a_\mu$ is not gauge-invariant meaning that the gauge group acts on the current field as

$$W_\mu \rightarrow U^{-1} W_\mu U. \quad (50)$$

However, the presence of the mass term of this field variable does not violate the gauge symmetry at all. This implies that the mechanism is fully independent of the gauge choice as needed. The question is that how the gauge-dependence of
the current field affect the theory? It will affect the physical interpretation of the theory since the quantity which is not gauge-invariant cannot be physical. On the one hand, in the Abelian gauge theory, the $W_\mu$ field is interpreted as the conserved current that is physically observable by experiment. On the other hand, in the non-Abelian gauge theory, the Noether’s current turns out to be not observable in practice. Mathematically, this fact is supported by the so-called **Weinberg-Witten theorem** [29] stating that the gauge theory with the spin is greater than $\frac{1}{2}$ cannot possess the gauge-invariant conserved current. Intuitively, it is understandable since, in the non-Abelian gauge theory, there is no concept of the current generated by the flows of the “free” quark.

Here again the decomposition as in the Abelian case can be done in the proper way, we can decompose the original gauge group $A_\mu$ into the current field $W_\mu$ and the remaining residual mode $R_\mu$. The way to understand the decomposition is to recap the historical concept called the **Cho-Faddeev-Niemi (CFN) decomposition** [3, 7, 11] which is the local version of the coset decomposition. The idea is as follows: in the beginning, the full gauge group is a Lie group $G$ which is mathematically associated with the Lie algebra $\mathfrak{g}$ through the Lie’s third theorem. The Lie algebra $\mathfrak{g}$ can be decomposed into the subalgebra $\mathfrak{h}$ and its coset $\mathfrak{c}$, i.e.

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{c}. \tag{51}$$

Suppose that the Lie algebra $\mathfrak{h}$ has the Lie group $H$ as its pair, the Lie group associated with the Lie algebra $\mathfrak{c}$ is $G/H$. Translating into the gauge field’s level, we can write

$$A_\mu = V_\mu + W_\mu, \tag{52}$$

where the massive mode $W_\mu$ belongs to the Lie algebra $\mathfrak{c}$ while the residual mode $V_\mu$ belongs to the rest. Since the residual mode transforms the same way as the original gauge group but with the different set of transformation, the presence of the residual gauge field’s mass is impossible in principle. This fact will be important in the next section below.
5 Implication in Confinement

To discuss the physical meaning of the gauge-independent Higgs mechanism in the non-Abelian gauge theory and understand it in the same way as mentioned in the literature, we will, from now on, call the $W_\mu$ as the off-diagonal gluon since it belongs to the coset space constructed from the full group modulo the diagonal transformation (Abelian). Interestingly, in the low energy quantum chromodynamics or IR-QCD for short, these massive modes decouple out from theory because the energy’s pool cannot excite these modes from nothing and the mass parameter $m_\nu$ that is generated through the studying mechanism plays a role as the energy scale telling how low the energy we talking about. The only remaining degree of freedom in the long-distance limit is the residual mode $V_\mu$ corresponding to the Abelian gauge group $H$. This fact is supported through the lattice simulation’s result \[19\] showing that the numerical value of the confined string tension is truly dominated by the Abelian contribution from the gauge field. Strictly speaking, this simply means that the gauge-invariant Higgs mechanism provides a possible mechanism to generate the mass of the gauge field dynamically which is used to understand the Abelian dominance phenomenon \[10\] in the QCD vacuum.

To see the more interesting result, we have to determine the explicit form of the residual gauge field $V_\mu$. To do so, we are meant to set the massive mode $W_\mu$ to be zero. Following the direct expression of the massive mode, setting the massive mode to be zero is equivalent to set the covariant derivative of the scalar field to be zero. In component, we have to solve for

$$\partial_\mu \hat{\Phi}^a + g f^{abc} V^b_\mu \hat{\Phi}^a = 0.$$  \hspace{1cm} (53)

Notice also here that if we set the massive mode to be zero, the full gauge group and the residual mode can be used interchangeably. Multiplying both sides by $f^{ade}$, using the identity \[46\] above, and then multiplying both sides also with $\hat{\Phi}^d$. We end up with

$$f^{ade} \hat{\Phi}^e \partial_\mu \hat{\Phi}^a + g V^d_\mu \hat{\Phi}^d \delta^{ce} \hat{\Phi}^a - g V^e_\mu \hat{\Phi}^e \hat{\Phi}^a = 0.$$  \hspace{1cm} (54)

After that, we choose the condition that $a = c$, it yields

$$f^{ade} \hat{\Phi}^d \partial_\mu \hat{\Phi}^e + g V^d_\mu \hat{\Phi}^d \hat{\Phi}^e - g V^e_\mu = 0.$$  \hspace{1cm} (55)
Writing back from the component from into the original expression, we can solve for the residual gauge field $V_\mu$ as

$$V_\mu = c_\mu \hat{\Phi}^a + \frac{i}{g} [\hat{\Phi}, \partial_\mu \hat{\Phi}],$$

where we have introduced the gauge-invariant quantity $c_\mu$ as

$$c_\mu \equiv A_\mu^a \hat{\Phi}^a.$$ (57)

The physical meaning of this new gauge-invariant quantity (57) will be clarified when the time is right. Since the Lie group that the residual gauge field belongs to is an Abelian $H$ group. It is very making sense to construct the curvature tensor from this gauge potential as

$$V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu.$$ (58)

Through the direct computation, the residual field strength can be written as the following expression

$$V_{\mu\nu} = \hat{\Phi} (\partial_\mu c_\nu - \partial_\nu c_\mu + H_{\mu\nu}),$$

where $H_{\mu\nu} \equiv \frac{i}{g} [\partial_\mu \hat{\Phi}, \partial_\nu \hat{\Phi}]$. It is expectable that the residual gauge field is directly proportional to the scalar field itself since the scalar field in the unitary gauge will align along the direction of the radial mode of the scalar field. The radial part of this field will be invariant under the stabilizer $H$ subgroup. It is also convenient to define the Abelian field strength $f_{\mu\nu}$ which is equivalently treated in the Lagrangian density level as

$$f_{\mu\nu} \equiv \hat{\Phi}^a V_{\mu\nu}^a = \partial_\mu c_\nu - \partial_\nu c_\mu + H_{\mu\nu}.$$ (60)

As we have mentioned earlier, in the long-distance or the low-energy regime, the massive gauge field $W_\mu$ can be integrated out from the theory, the only remaining contribution is dominated by the residual gauge mode, the low-energy effective Lagrangian becomes the only

$$\mathcal{L} = -\frac{1}{4} V_{\mu\nu} V_{\mu\nu}^a = -\frac{1}{4} f_{\mu\nu} f_{\mu\nu}.$$ (61)

This Lagrangian, in the literature, is known as the restricted Yang-Mills theory encoding the Abelian dominance feature of the low-energy approximating Yang-Mills model. Now we are ready to discuss the physical interpretation of the field $c_\mu$ and
the new curvature tensor $H_{\mu\nu}$. Observe that we compute the dual Abelianized field strength tensor through Hodge duality transformation in the 4-dimensional manifold as

$$\tilde{f}^{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} f^{\alpha\beta}. \quad (62)$$

Observe that the first-half contribution or the contribution from $c_{\mu}$ field satisfies the Bianchi identity $\partial_\nu \tilde{f}^{\mu\nu} = 0$ identically, namely, we can show that

$$\frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \partial_\nu (\partial_\alpha c_\beta - \partial_\beta c_\alpha) = \varepsilon^{\mu\nu\alpha\beta} \partial_\nu \partial_\alpha c_\beta = -\varepsilon^{\mu\nu\alpha\beta} \partial_\alpha \partial_\nu c_\beta = 0. \quad (63)$$

However, there is no guarantee that the last-half contribution or the $H_{\mu\nu}$ dependence will not violate the Bianchi identity. If we compute with the modified version of Maxwell’s equation, the $c_{\mu}$ will play a role as the Coulomb-like electric current of the considering model while the $H_{\mu\nu}$ that is meant to violate the Bianchi identity in the generalized Maxwell’s equation is represented as the magnetic monopole contribution. Thus, this mechanism does not only encode the Abelian dominance behavior of the Yang-Mills theory but also show that the Abelianized potential can be further decomposed into the photon-like and the magnetic monopole like contribution. This is also supported by the numerical lattice calculation in the sense that the lattice result suggests that the compact $U(1)$ gauge theory [2, 6] possesses two phases: the confined phase in the strong coupling region [1, 25] generated by the formation of the magnetic flux tube in the magnetic monopole condensate and the Coulomb phase [13, 16] in the weakly coupled regime where the photon-like gauge field dominates the phase.

At the end of this section, we will remark that the gauge-independence Higgs mechanism provides very strong evidence of the mass gap $m_W^2 \approx \langle W^\mu W_\mu \rangle$ in the Yang-Mills-Higgs theory can be generated in a dynamical way and the presence of the mass condensation implies also the presence of the confinement-deconfinement phase transition. Observe that the confined behavior we have discussed relies on the absence of the massive mode in the low energy theory. Thus, the mass of the massive gauge field $m_W$ can be treated approximately as the thermodynamic scale separating the confinement and the deconfinement phase in the QCD phase diagram. Numerically, we can compute the mass of the massive gluon mode through
the lattice computation \cite{27} as

\[ m_W \approx 1.19 \text{ GeV}. \quad (64) \]

\section{Reduction Condition}

Now we arrive at the final topic of this review article, as we have seen in the previous section that the gauge-independence Higgs mechanism is a promising mechanism to describe the confinement behavior of the “Yang-Mills-Higgs model”. However, in the real world, the confinement behavior occurs even in the pure gluon sector (pure Yang-Mills theory). In particular, there is no at all a scalar field or the Higgs field in the QCD in the first place. Due to this reasoning, we have to find a way to eliminate the extra degrees of freedom of the scalar field. The way to do so was suggested in the context of the CFN decomposition (see also \cite{21}). In this CFN representation of the Yang-Mills model is suffered from the additional gauge transformation. Following this, we enlarge the gauge group of the Yang-Mills-Higgs model into the higher gauge group controlled by two gauge parameter, i.e.

\[ G^\alpha \times G/H^\beta, \quad (65) \]

where \( G \) is the original isometry group that the original Yang-Mills theory experiences. This infinitesimal gauge group action on the full gauge potential takes the form

\[ \delta_\alpha A_\mu = \frac{i}{g} D_\mu \alpha. \quad (66) \]

The verification is pretty easy by substituting the element of the gauge group with full expression \( U = e^{i\alpha} \), performing the Taylor expansion by requiring that the gauge parameter \( \alpha \) is infinitesimally small, and finally remembering that the gauge parameter \( \alpha \) is not commutative with the gauge field. On the other hand, the extra gauge symmetry is described by the gauge group \( G/H^\beta \) acting on the scalar field as

\[ \delta_\beta \hat{\Phi} = i[\hat{\Phi}, \beta]. \quad (67) \]

Here we can use these two transformations to find out how they act on the massive vector field \( W_\mu \). Since the gauge field that controls the behavior of the coset transformation is the massive mode itself. Thus, we just have to fix an additional gauge
degree of freedom in such a way that minimizes the Hilbert-Schmidt norm of the massive vector field. After performing the minimization, we will obtain the appropriate constraint to fix the extra degree of freedom. This gauge fixing condition can be found to be \[ \chi \equiv [\hat{\Phi}, D^\mu D_\mu \hat{\Phi}] = 0, \] (68)

where, from now on, we will call this gauge condition as the reduction condition. The reason for the name will be understood properly in the following manner. It can be shown further that the reduction constraint is completely orthogonal to the radial direction of the scalar field

\[ \chi^a \hat{\Phi}^a = 0. \] (69)

This implies that the gauge condition can eliminate the degree of freedom created by the presence of the scalar field. Speaking strictly, with this appropriate reduction condition, the Yang-Mills-Higgs model will reduce precisely into the pure Yang-Mills theory with the right degrees of freedom. Thus everything we have done earlier can be eventually translated into the language of the pure Yang-Mills as we needed.

7 Conclusion

In this review article, we firstly show why we need this mechanism, to be precise, what is the blind spot of the spontaneous symmetry breaking in the gauge theory. Next, we have reviewed the idea of the Higgs mechanism without relying on the aspect of the spontaneous symmetry breaking. We have discussed the physical interpretation of the gauge-independent Higgs mechanism in both the Abelian gauge model and the non-Abelian one. We will finally study the way to reduce the Yang-Mills-Higgs model into the real world gluonic QCD reference model, the pure Yang-Mills sector.

References


